

# Baryonic hybrids: Gluons as beads on strings between quarks

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In this paper we analyze the ground state of the heavy-quark  $qqqG$  system using standard principles of quark confinement and massive constituent gluons as established in the center-vortex picture. The known string tension  $K_F$  and approximately-known gluon mass  $M$  lead to a precise specification of the long-range non-relativistic part of the potential binding the gluon to the quarks with no undetermined phenomenological parameters, in the limit of large interquark separation  $R$ . Our major tool (also used earlier by Simonov) is the use of proper-time methods to describe gluon propagation within the quark system, along with some elementary group theory describing the gluon Wilson line as a composite of co-located  $q$  and  $\bar{q}$  lines. We show that (aside from color-Coulomb and similar terms) the gluon potential energy in the presence of quarks is accurately described (for small gluon fluctuations) via attaching these three strings to the gluon, which in equilibrium sits at the Steiner point of the Y-shaped string network joining the three quarks. The gluon undergoes small harmonic fluctuations that slightly stretch these strings and quasi-confine the gluon to the neighborhood of the Steiner point. To describe non-relativistic ground-state gluonic fluctuations at large  $R$  we use the Schrödinger equation, ignoring mixing with  $l = 2$  states. Available lattice data and real-world hybrids require consideration of  $R$  values small enough for significant relativistic corrections, which we apply using a variational principle for the relativistic harmonic oscillator. We also consider the role of color-Coulomb contributions. In terms of interquark separations  $R$ , we find leading non-relativistic large- $R$  terms in the gluon excitation energy of the form  $\varepsilon(R) \rightarrow M + \xi[K_F/(MR)]^{1/2} - \zeta\alpha_c/R$  where  $\xi, \zeta$  are calculable numerical coefficients and  $\alpha_c \simeq 0.15$  is the color-Coulomb  $q\bar{q}$  coupling. When the gluon is relativistic,  $\varepsilon \sim (K_F/R)^{1/3}$ . We get an acceptable fit to lattice data with  $M = 500$  MeV. Although we do not consider it in full detail, we show that in the  $qqG$  hybrid the gluon is a bead that can slide without friction on a string joining the  $q$  and  $\bar{q}$ . We comment briefly on the significance of our findings to fluctuations of the minimal surface, a subject difficult to understand from the point of view of center vortices.

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## I. INTRODUCTION

In QCD, hybrids are hadrons that cannot be understood as combinations of quarks and antiquarks only, but that require consideration of gluonic excitations along with the quarks. Perhaps the most interesting of these are the exotic hybrids, which are hadrons whose quantum numbers are not found in any colorless combination of  $q$  and  $\bar{q}$ , and so must contain valence gluons. Hybrids and exotics [1] have been studied theoretically for decades (for example, [2], [3]), but these early studies are tied to models and approximations that hardly make it possible to choose one model from another. Experiments have searched for hybrids as well, and there are a number of credible observed candidates for mesonic hybrids (for a recent review see [4]). The author knows of no plausible candidates for baryonic hybrids.

Unfortunately, theoretical approaches to hybrids are, naturally enough, even more forbidding than approaches to ordinary hadrons. So one might wonder what the value of a theoretical paper on hybrids is. The answer is that recent lattice simulations [5, 6] of heavy static quarks (described by straight Wilson lines in the Euclidean time direction) with single-gluon excitations provide a valuable testing-ground for theoretical ideas, because one does not have to deal with the extra complication of quark Wilson-line fluctuations and because lattice data are available at theoretically-interesting separations of quarks.

We will be primarily interested in estimating the ground-state gluonic excitation energy  $\varepsilon(R)$  of the heavy-quark baryonic hybrid  $qqqG$ , as a function of a suitably-defined interquark separation  $R$ , and in comparing our results to lattice data [5]. We will also remark in less detail on similar issues for mesonic hybrids, but will not attempt to compare to data in this paper. The regime where we are on firmest ground is the regime of asymptotically-large large

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$R$ , where the dynamics is non-relativistic (because the gluon has a dynamical mass) and we can fairly confidently use the Schrödinger equation. However, lattice data [5] are available for smaller  $R$ , where relativistic corrections are important, and to cover the entire relevant range of  $R$  we use a variational principle incorporating fully-relativistic kinetic energy for the gluon. The smallness parameter for non-relativistic motion is a numerical constant times  $[K_F/(M^3R)]^{1/2}$ , where  $K_F$  is the string tension and  $R$  the interquark separation. It turns out that our calculations are in excellent agreement with the lattice data for a gluon mass of 500 MeV, even for  $R$  small enough for relativity to matter.

One should note that in the fully-relativistic regime there should be mixing with other states containing more gluons, such as baryon plus glueball. Additionally, the derived potentials have angular momentum  $l = 2$  terms as well as  $l = 0$ . These lead to mixing of gluonic  $l = 2$  and  $l = 0$  states, and should lower the energy slightly compared to the results we give. This lowering might be compensated by raising the gluon mass. We do not consider either of these mixings here.

The idea that the gluon has a constituent mass  $M$  [7] goes back a long way, but a purely theoretical determination of  $M$  is not yet within our powers. Lattice simulations have verified this idea and found values for the mass, although many such simulations, done in particular gauges, cannot guarantee a gauge-invariant determination of the mass. However, Ref. [8] has extrapolated their Euclidean lattice data to the Minkowski regime, to find the so-called pole mass, which is gauge-invariant. The pole mass so found on the lattice is  $M \simeq 600 (+150 -30)$  MeV [8]. We believe that our fitted value of 500 MeV is acceptably close to the lattice pole mass, given the expected uncertainties in our approach and in the lattice determinations.

There are two main steps in finding the dependence of the gluonic energy on  $R$ . The first uses a proper-time technique for describing propagation of the gluon in the hybrid, and the second, as described above, implements the dynamical fluctuations in the proper-time gluon paths with a Schrödinger equation modified for relativistic effects. In the first step, we describe the propagator of the gluon in the heavy-quark hybrid as a quantum perturbation on a background of confining gauge potentials. The center-vortex picture of confinement prescribes [9, 10] that the quarks are joined by a Y-shaped network of minimal surfaces (or strings, at fixed time). The gluon propagator has a proper-time form that is an integral over all paths of an adjoint Wilson line going along a path joined at both ends to the system of heavy-quark Wilson lines. [These techniques were developed earlier by Simonov [11] and applied to  $q\bar{q}G$  hybrids, but not  $qqqG$  hybrids. Simonov's works were unknown to the present author, who developed them independently and discovered Simonov's work after this paper was finished.] As we will see, the result is that if we do not distinguish the three quarks by flavor or other quantum numbers, the gluon can be described as a massive bead attached to the three strings joining the quarks in the hybrid, and fluctuating in directions transverse to the string. The fluctuations stretch the string slightly, and there is a harmonic restoring force on the gluonic bead. In the  $q\bar{q}G$  hybrid the gluon is a bead sliding without friction on the string joining the quark and antiquark.

All the potentials we use in the second step are harmonic, found by expansion of various minimal areas occurring in the baryonic Wilson loop. Except in the ultra-relativistic limit and the Newtonian limit this relativistic Schrödinger equation is not analytically solvable, but good approximations to  $\varepsilon(R)$  for large  $R$  can be found by a variational technique. We describe this technique in the Appendix. For the  $qqqG$  system, the color-Coulomb energy can be straightforwardly estimated classically and added as a perturbation. It is not quite so straightforward for the  $q\bar{q}G$  hybrid color-Coulomb energy, which we describe in the Appendix. The Appendix also treats the string energy for the  $q\bar{q}G$  hybrid, finding relativistic gluon energies whose leading term is of the form  $N\pi/R$ . Lattice data on heavy-quark  $q\bar{q}G$  systems suggest such leading string-like energies, but with substantial corrections [6, 12]. These corrections can be estimated with our techniques, but we do not study the problem here.

Our final result yields three terms in the large- $R$  (non-relativistic) expansion of the  $qqqG$  energy:

$$\varepsilon(R) \simeq M + \xi \left( \frac{K_F}{MR} \right)^{1/2} - \frac{\zeta \alpha_c}{R} + \dots \quad (1)$$

where  $\xi, \zeta$  are numerical parameters that we estimate with fair accuracy below and in the Appendix, and  $\alpha_c$  is the coefficient of  $-1/R$  in the  $q\bar{q}$  Coulomb energy. At shorter distances, where relativistic corrections are important, the gluon energy scales like  $(K_F/R)^{1/3}$ , and Coulomb corrections are smeared out by the gluon wave function, as discussed in the Appendix.

There are numerous other corrections to the small- $R$  results, including spin-dependent terms and poorly-understood terms that arise [9] when two Wilson loops partly coincide, as is the case for quark Wilson loops formed from the gluon propagator. We ignore all such terms, which do not seem to be very large, judged by our fit to the lattice data. It is somewhat surprising that our techniques give good results even for fairly short distances, considering all the possible effects that can enter.

## II. PROPER-TIME TECHNIQUES AND WILSON LOOPS

In gauge-theory dynamics, a Wilson line or loop is to be integrated not only over all gauge configurations but also over all possible lines joining the fixed endpoints, with a certain weight function. This weight function encodes all the particulars about the mass, flavor, and spin of the particle being described by the Wilson line. In other words, particle propagation is to be described by a proper-time path integral.

### A. Gluon proper-time propagator

We wish to construct the necessary path integral for a gluon in a hybrid state, where distance scales are large compared to the QCD scale. We will argue below that gluonic spin contributes only short-range effects, which allows us to simplify matters by considering the propagation of a scalar particle in the adjoint representation. In a fixed background gauge potential the Euclidean proper-time propagator  $\Delta(x; y)$  from  $x$  to  $y$  of a scalar particle of mass  $M$  is

$$\Delta(x; y) = \mathcal{N} \int_0^\infty \int_x^y (dz) \exp\left\{\frac{-M}{2} \int_0^s d\tau [\dot{z}^2 + 1]\right\} U_A(x; y) \quad (2)$$

where  $(dz)$  stands for the integral over all paths  $z_\mu(\tau)$  from  $x$  at  $\tau=0$  to  $y$  at  $\tau=s$ ;  $\mathcal{N}$  is a normalization factor; and  $U_A(x; y)$  is the adjoint Wilson line

$$U_A(x; y) = P \exp\left[ig \int d\tau \dot{z}_\mu T^\alpha A_\mu^\alpha(z)\right]. \quad (3)$$

Here  $T^\alpha$  are the adjoint generators, and  $P$  stands for path-ordering. Of course, the propagator  $\Delta$  is not gauge-invariant by itself, but we will soon couple it to quarks in a gauge-invariant way.

If (still ignoring spin) we identify  $\Delta$  as the propagator of the potential  $A_\mu^\alpha(z)$  that occurs in  $U_A$ , intractable problems arise. So we envision the gauge potential as being separated into a confining part (condensate of center vortices) that constitutes the background potential and the small-amplitude part whose propagation we are describing; it is this latter part that constitutes the gluonic excitation in a hybrid hadron. As for spin itself, the propagation of a vector gauge boson with small amplitude in a background field requires that  $\dot{z}_\mu$  in  $U_A$  be replaced by  $\dot{z}_\mu + \Sigma_{\mu\nu} \partial_\nu$ , where  $\Sigma_{\mu\nu} \partial_\nu$  are the spin-one generators of rotations in Euclidean four-space, and path ordering is also carried out with respect to the spin operator. However, the derivative yields the background potential field strength, which is short-ranged. Since our primary interest is in long-range effects, we will from now on ignore gluon spin, and think in terms of the explicit scalar propagator of Eq. (2).

### B. Splitting the adjoint Wilson line

It is familiar fact that a gluon line in a Feynman diagram can be replaced by co-located quark and antiquark lines, joined to other quarks according to certain rules. The usual large- $N$  counting rules, for example, follow from such considerations. Perhaps less familiar is the dynamical application of this splitting of a gluon line [11], which we now describe. The resulting picture is most easily described for  $q\bar{q}G$  hybrids, which we discuss first.

#### 1. Mesonic hybrids

Let

$$H_\nu(x) = \bar{\psi} \gamma_\mu \frac{1}{2} \lambda^\alpha \psi G_{\mu\nu}^\alpha(x) \quad (4)$$

be the interpolating field for a hybrid meson, with  $(1/2)\lambda_\alpha$  the conventionally-normalized generator in the fundamental representation. (It happens to be the field describing an exotic, with  $J^{PC} = 1^{-+}$  [2].) The field  $\psi$  describes very massive quarks. Ignoring spin indices and other irrelevant complications, the  $H$ -field propagator  $\Delta_H$  is an expectation value

$$\Delta_H = \langle U_l^i(1) U_j^k(2) [U_A(g)]_{ik}^{jl} \rangle \quad (5)$$

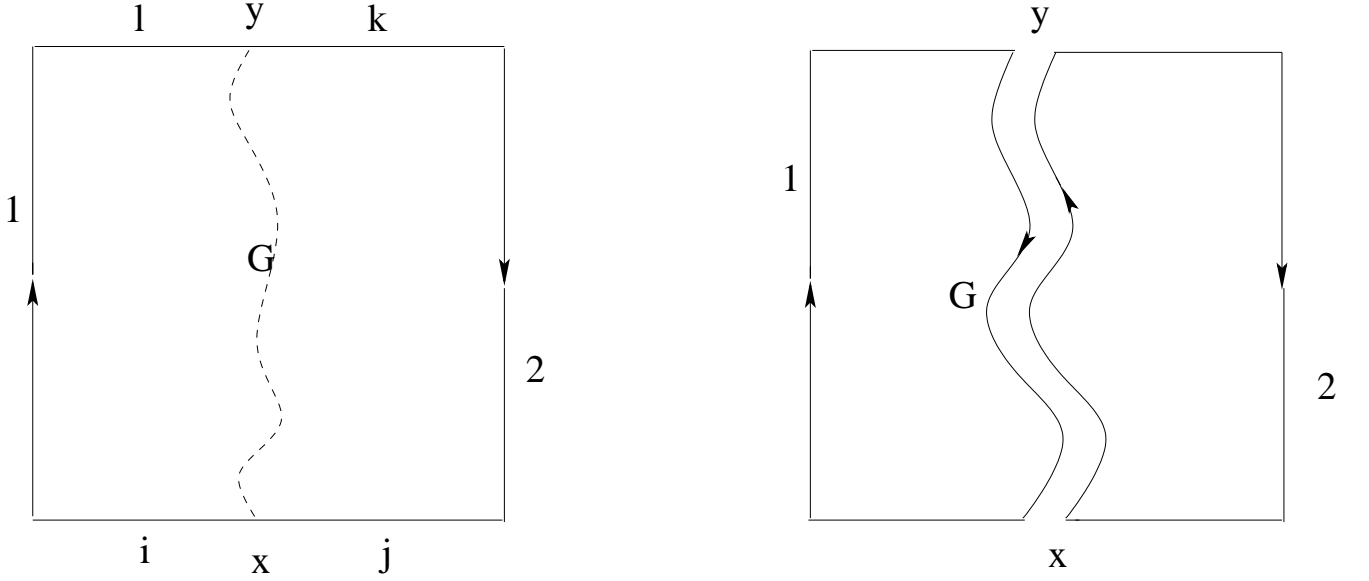


FIG. 1: Left: A rectangular Wilson loop for a quark (1) and antiquark (2) and a gluon (G). Right: The same loop with the gluon line decomposed into another quark and antiquark line, forming two standard Wilson loops.

where  $U(1, 2)$  are fundamental Wilson lines for the quark and antiquark, and  $\langle \cdot \rangle$  means to integrate over all gauge potentials and over the gluon Wilson line as shown in Eq. (2). The gluon Wilson line matrix has been converted into a matrix with fundamental indices on it, via

$$[U_A(g)]_{ik}^{jl} = \frac{1}{2}(\lambda_\alpha)_i^j (\lambda_\beta)_k^l [U_A(g)]_{\alpha\beta}. \quad (6)$$

Here  $U_A$  is the adjoint representative of the group element  $g$ , which is the path-ordered product seen in Eq. (3). We show the index structure of Eqs. (5, 6) in Fig. 1.

Now we split the gluon line, using the  $SU(N)$  equations

$$\frac{1}{2}(\lambda_\alpha)_i^j (\lambda_\beta)_k^l = \delta_k^j \delta_i^l - \frac{1}{N} \delta_i^j \delta_k^l \quad (7)$$

and

$$[U_A(g)]_{\alpha\beta} = \frac{1}{2} \text{Tr}[U(g) \lambda_\alpha U^\dagger(g) \lambda_\beta]. \quad (8)$$

Combining leads to a familiar equation:

$$[U_A(g)]_{ik}^{jl} = U_i^l(g) [U^\dagger(g)]_k^j - \frac{1}{N} \delta_i^j \delta_k^l. \quad (9)$$

This equation has the group property: If  $g = g_1 g_2$ , the product

$$[U_A(g_1)]_{ib}^{ja} [U_A(g_2)]_{ak}^{bl}, \quad (10)$$

when computed using the right-hand side of Eq. (9), has the form of that right-hand side with  $g$  replaced by  $g_1 g_2$ .

For us,  $g$  is the group element in the gluon Wilson line [see Eq. (3)]. The adjoint on the right-hand side of Eq. (9) means to trace the path backwards, as one easily checks. Therefore the hybrid propagator in Eq. (5) assumes the simple form

$$\Delta_H(x; y) = \langle \text{Tr} U(1G) \text{Tr} U(G2) \rangle + \dots \quad (11)$$

where  $1G$  is the closed fundamental Wilson loop shown in the right side of Fig. 1, and similarly for the other closed loop  $G2$ . The omitted term comes from the  $1/N$  term in Eq. (9), and yields a single Wilson loop describing a  $q\bar{q}$  pair. We will drop this term, which in any case cannot contribute for exotic hybrids.

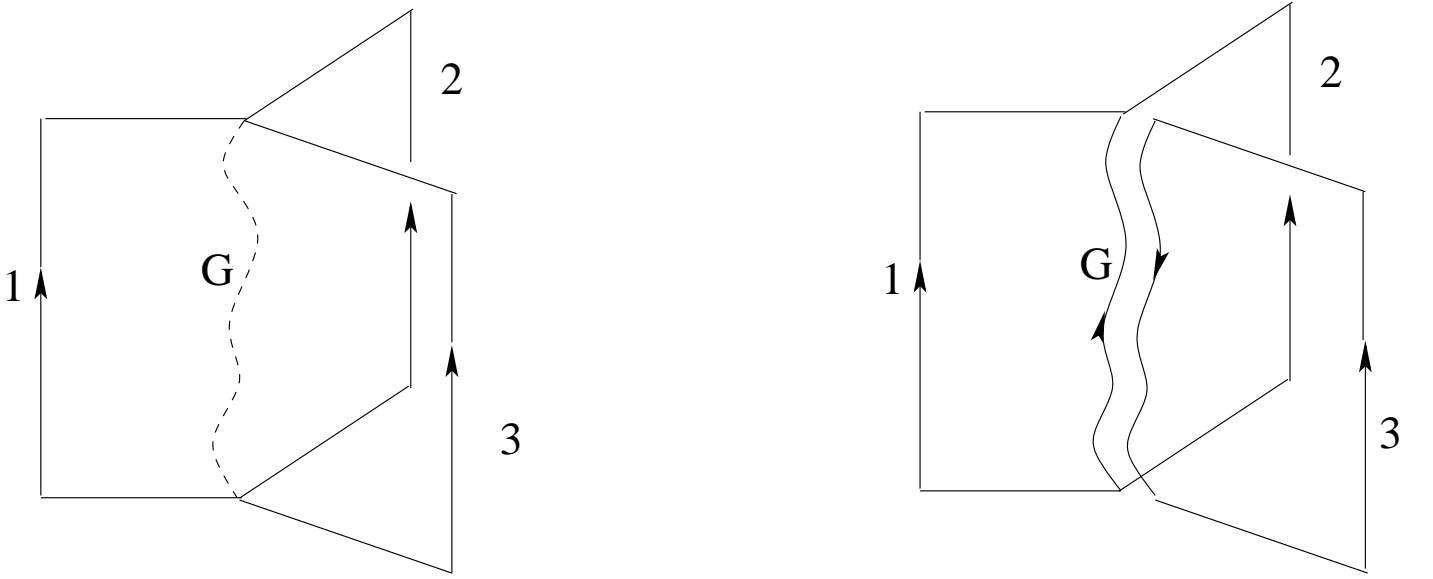


FIG. 2: Left: Wilson loop for a  $qqqG$  configuration (quarks labeled 1, 2, 3; gluon labeled G). Right: The same loop with the gluon line decomposed into another quark and antiquark line, with quark line 3 singled out as discussed in the text.

## 2. Baryonic hybrids

Similar considerations apply for  $qqqG$  hybrids, but there are minor group-theoretic complications. The direct product of three quark representations is

$$3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8' \oplus 10 \quad (12)$$

There are two 8s which can couple to a gluon. Denote the quark group vectors by  $a^i, b^k, c^l$ , and form the octet combination

$$V_j^i(a; b, c) = \epsilon_{jkl} a^i b^k c^l - \frac{1}{3} \delta_j^i \epsilon_{pqr} a^p b^q c^r. \quad (13)$$

Quark  $a$  has been singled out (its index is external). By permuting the quark group vectors one seems to come to three possible octets, but because of

$$V_j^i(a; b, c) + V_j^i(b; a, c) + V_j^i(c; a, b) \equiv 0 \quad (14)$$

there are really only two.

We can construct a propagator for a  $qqqG$  hybrid in the same spirit as the earlier work for the mesonic hybrid. Let there be three heavy-quark lines labeled 1, 2, and 3, plus gluon line  $G$ . We write the propagator for one of the two octets, singling out line 3 as we did quark  $a$  above (again dropping the trace term, which is missing the gluon):

$$\langle \epsilon_{abc} \epsilon^{pqr} U_p^a(1) U_q^b(2) U_j^i(3) U_A(g)_{ir}^{cj} \rangle. \quad (15)$$

Using the identity of Eq. (9) as before yields the  $qqqG$  analog of the gluon line splitting for mesons as shown graphically in Fig. 2. Line 3, which has been singled out, forms a  $q\bar{q}$  state with one of the split gluon lines, while the other split gluon line forms a new baryon  $12G$ .

## C. Gluon dynamics

In the quark Wilson loops of Figs. 1, 2, the rectangular lines do not fluctuate (the quarks are infinitely heavy). But the lines associated with the gluon do fluctuate, since the gluon path-integral weight function has a finite mass. The dynamics of the quark loops is governed by conventional area laws, and we now turn to the question of how those area laws govern the gluon dynamics.

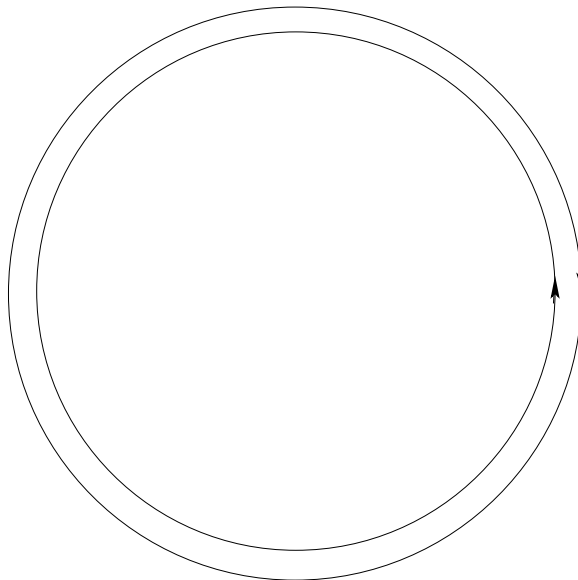


FIG. 3: An adjoint Wilson loop displayed as the superposition of two oppositely-oriented Wilson loops.

The center-vortex picture of the area laws for a fundamental Wilson loop claims [9] that, without the extra gluonic effects of the present paper, the area to be used in the area law is the minimal area spanning the given Wilson loop. (This only makes sense if there is a unique such area, which is certainly true in the present case; see [9] for further discussion.)

#### D. Qualitative behavior of gluonic potentials

Let us begin with some qualitative remarks. Consider a Wilson loop in the adjoint representation, which we decompose into two fundamental (quark) Wilson loops in the now-familiar way as shown in Fig. 3. Because the superposed loops are oppositely-oriented, the linkage of a center vortex with one loop is exactly and coherently cancelled by the linkage of the same vortex to the other (oppositely-oriented) loop. Thus the center-vortex linkage contributes a factor of unity to the adjoint Wilson loop, as must be the case.

Consider the right-hand part of Fig. 1, showing the gluon world line decomposed as a quark and antiquark line. The area law for the original  $q\bar{q}$  pair, labeled 1 and 2, is the sum of the two areas on the right (which of course is the original area on the left side of the figure), as long as the gluon line stays in the original minimal surface (plane of the figure) and does not stray too close to the lines 1 and 2. (As the gluon line approaches these lines, perimeter terms arise because one or the other of the two Wilson loops in the right-hand part of the figure has a length scale comparable to the QCD scale). That is, fluctuations of the gluon line which are confined to the original minimal surface change nothing; the gluon does not feel the  $q\bar{q}$  pair.

If the gluon line fluctuates to a position not on the original minimal surface, there are forces acting to pull it back. Just as for the adjoint Wilson loop of Fig. 3, these are not confining potentials (area laws), but are due to perimeter and other forces, including color-Coulomb forces. Consider first a gross deformation of the Wilson loop of Fig. 1 (right), as shown in Fig. 4. Although the areas enclosed by contours 1 and 2 have increased, just as for Fig. 1 they are oppositely-oriented. The only area law refers to the original 1-2 loop. In fact, if one tries to pull the gluon a long way from the  $q\bar{q}$  loop, the adjoint string discussed above will break, forming a gluon pair, and the result will be a hybrid plus a glueball.

Evidently the techniques described here are applicable to the problem of fluctuations of the area in a confining area law, although this fluctuation problem is more complicated because one must treat an arbitrary number of gluons, not part of the condensate of center vortices, exchanged between parts of the Wilson line. Originally, area-law surface fluctuations were thought of as taking place in chromoelectric flux tubes [13]. However, the center vortex description of confinement is different, and requires a new approach. We discuss this a little more in the concluding section.

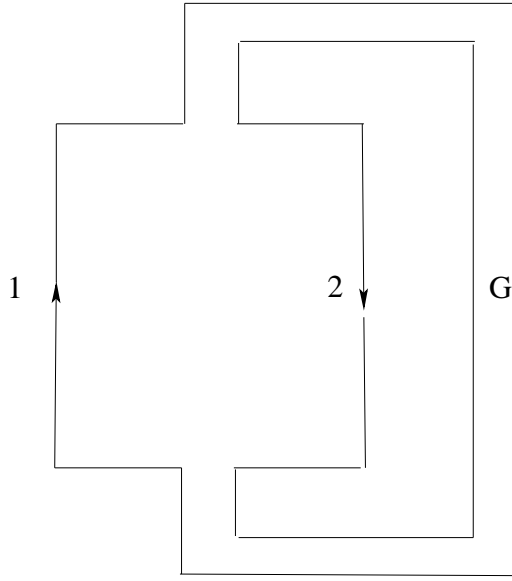


FIG. 4: The gluon, decomposed as a  $q\bar{q}$  pair, is pulled far from the original minimal surface of Fig. 1.

### E. Toward quantitative gluonic potentials

If the two Wilson loops  $1G$  and  $G2$  shown in the right side of Fig. 1 were statistically independent with respect to the center vortices linking them, the expectation value in Eq. (11 for the hybrid propagator would factor, and we would have for this propagator

$$\begin{aligned} \Delta_H(x; y) &= \mathcal{N} \int_0^\infty \int_x^y (dz) \exp\left\{\frac{-M}{2} \int_0^s d\tau [\dot{z}^2 + 1]\right\} \langle \text{Tr} U(1G) \rangle_A \langle \text{Tr} U(G2) \rangle_A \\ &= \mathcal{N} \int_0^\infty \int_x^y (dz) \exp\left\{\frac{-M}{2} \int_0^s d\tau [\dot{z}^2 + 1]\right\} \exp[-K_F(A(1G) + A(G2))] \end{aligned} \quad (16)$$

where  $\langle \cdot \rangle_A$  is the expectation value over center vortex configurations and we have explicitly exhibited the gluonic path integral. In this equation  $A(1G)$  is the minimal surface spanning contour  $1G$ . Actually, the center vortex linkages in the two loops are not quite independent, because the loops share the leg  $G$ , but the correlation effects are short-ranged (see [9] for a discussion) and will be ignored here.

Take the original Wilson loop  $12$  of Fig. 1 to be a rectangle of length  $T$  in the Euclidean time direction  $t$ , and of length  $R$  in the  $z$  direction, centered at  $z = 0$ . We will extract the non-relativistic potential from the path integral of Eq. (16) as the coefficient of the length  $T$  of the time-directed segments of the original quarks paths  $1, 2$ , in the limit  $\rightarrow \infty$ . As mentioned previously, all potentials for hybrids will be expressed relative to the potential in the original  $q\bar{q}$  system, so we subtract  $RT$  from the area  $A(1G) + A(G2)$  in determining the gluon potential. If the gluonic contour segment  $G$  lies entirely in the original (flat)  $q\bar{q}$  plane, then the spanning surfaces remain flat and the sum of areas  $A(1G) + A(G2)$  is  $RT$ , so the potential is zero. But for small transverse excursions of  $G$  into the  $x, y$  directions, the sum of areas is slightly increased. The simplest formula for this small change in the minimal surface area involves presenting the coordinates perpendicular to the surface, which we take to be  $x, y$ , as harmonic functions of two surface variables, which we take to be  $z, t$ :

$$A(1G) + A(G2) - RT \simeq \frac{1}{2} \int_{\Gamma(1G) + \Gamma(G2)} dt dz [x'^2 + \dot{x}^2 + y'^2 + \dot{y}^2] \quad (17)$$

where the dot indicates a  $t$  derivative, the prime indicates a  $z$  derivative, and the integral is over the flat surfaces  $\Gamma(1G)$ ,  $\Gamma(G2)$  spanned by the original quark lines and the projection of the gluon contour  $G$  onto  $x, y = 0$ . At first sight this is confusing, because the gluonic path variables are originally presented as a Euclidean four-vector  $z_\mu(\tau) = (x, y, z, t)$ , with all four coordinates depending on the proper-time variable  $\tau$ . Because the gluon is assumed non-relativistic, Euclidean time  $t$  is essentially the same as proper time  $\tau$ . And the functions  $x(z, t), y(z, t)$  required in the area integral of Eq. (17) are harmonic and therefore, in principle at least, determined by their boundary value on the contour  $z_\mu(\tau)$ .

Note that the two transverse variables  $x, y$  are just the variables used in [13] to describe fluctuations of a surface formed from a chromoelectric flux tube extended in time. There should be, then, a close relation between the dynamics of a valence gluon in a hybrid and surface fluctuations. We will not explore that relationship further here, but it is a very interesting subject.

Next, we assume that the actual gluon paths  $z_\mu(\tau)$  stay close to the original minimal surface and are not highly-convoluted. In that case, the perturbed surface is nearly flat. An example is a portion of the helicoid, a minimal surface generated by rotating a rigid straight line segment around a fixed axis while at the same time moving it along the axis. Let the axis be the segment of the quark line 1 in the  $t$  direction, and the line segment at  $t = 0$  is the  $z$  axis from the origin to line 1. This line segment has length  $R/2$ . The line segment when  $t$  has a value scaling with the overall time  $T$  is slightly rotated around line 1 so that it passes through the point  $x = 0, z = 0, y$  with  $y \ll R/2$ . The slightly-curved contour  $z_\mu(\tau)$  is implicit in this description, and we need not be explicit; it is enough to observe that only small errors are made by approximating all portions of the contours by straight lines. Because the contour goes from  $x, y = 0$  at  $t = 0$  to small values  $x, y$  at  $t \sim T$ , it is clear that  $\dot{x} \sim x/T$ ,  $\dot{y} \sim y/T$ . Since  $T \rightarrow \infty$  we can drop the time derivatives. By a similar line of argument,  $x' \sim x/R$ ,  $y' \sim y/R$ . Then the area integral in Eq.(17) is of the form

$$A(1G) + A(G2) - RT \sim T \left[ \frac{1}{2R} (x^2 + y^2) \right]. \quad (18)$$

We need not work out the full details, because it should be clear by now that if we approximate pieces of the full contour by straight lines we are simply describing physically the gluon as a bead that runs without friction along a string stretched from  $z = R/2$  to  $z = -R/2$ . Transverse string fluctuations stretch the string and give rise to the potential of Eq. (19) below. As is evident from Eq. (18), the small-fluctuation potential is that of a harmonic oscillator.

Before discussing this potential (and its  $qqqG$  analog) we return briefly to the perimeter terms that keep the gluon from going too far beyond the original contour 12. These perimeter terms in the adjoint Wilson loop are discussed in the center-vortex picture in [14],[15]. These terms give rise to a breakable string: A potential which is approximately linearly-rising, but goes flat when enough energy is stored in the string to pop a gluon pair out of the vacuum. It has been argued (see [14] and references therein) that  $K_A$  is related to  $K_F$  by Casimir scaling; another elementary argument is that  $K_A = 2K_F$  because an adjoint string is two fundamental strings. The rough numerical estimates of [15] yield a  $K_A$  that is bigger than  $K_F$ , but not quite according to either of these arguments. In the present work it is good enough, within the context of our approximations, to take  $K_A = 2K_F$ .

Note that the adjoint string necessarily breaks when an energy comparable to  $2M$  is stored in it. This means that the more relativistic the gluon, the closer one approaches to string breaking and mixing of the hybrid with states containing another gluon. We will not take this issue up here.

### III. GLUONIC POTENTIALS

Rather than continue to deal directly with fluctuations of a minimal surface, as in Eq. (18), we will find a very good approximation to the potentials we need by starting from linearly-rising potentials attaching the gluon to the quark lines. Of course, these potentials cannot be taken literally when the gluon fluctuates far from the original minimal surface, and only their expansions for small fluctuations are of interest. As we saw in the last section, these give rise to harmonic oscillators with frequency  $\sim [K_F/(MR)]^{1/2}$ . Small fluctuations mean small gluon velocities, and so this regime of small fluctuations should be non-relativistic, if the gluon mass is large enough.

For the  $q\bar{q}G$  hybrid the gluon can move freely along the string, except that it cannot pass beyond the string endpoints. The free longitudinal momentum is associated with a momentum  $p$  in this surface of the form  $p = N\pi/R$ . If  $N > 1$  the motion is relativistic for  $R$  is less than about 2 fm, and so we will concern ourselves only with the ground state  $N = 1$ . Details of an approximate non-relativistic analysis of the combined longitudinal and transverse Schrödinger equation are in the Appendix.

As we will see, the numbers turn out such that even for the gluonic ground state of a hybrid relativistic corrections seem to be important, and we must ask where and how relativistic effects enter. We find in the Appendix that the gluon velocity  $\beta$  obeys  $\beta^2 \sim [K_F/(M^3R)]^{1/2}$ , and relativistic effects are important for  $R \simeq 0.6$  fm.

Although we do not study mesonic hybrids in any detail in this paper, it is worth giving the general scheme of the gluonic potential for the  $q\bar{q}G$  system, both for future use and because the same themes come up for the  $qqqG$  potential.



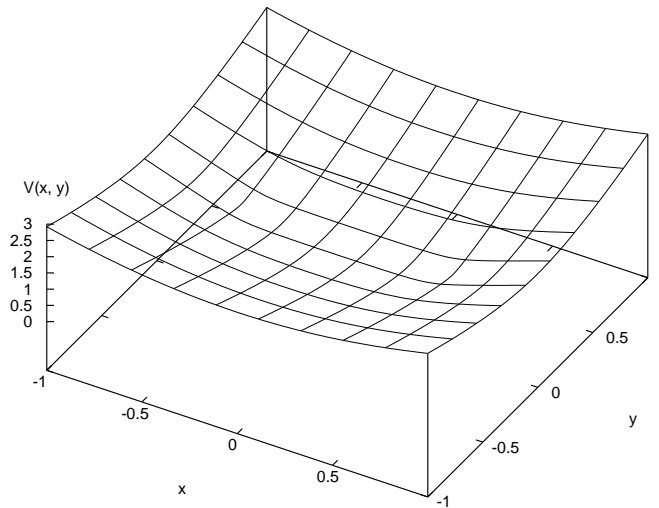


FIG. 5: The potential of Eq. (19), with  $R$  and  $K_F$  scaled to unity. The  $y=0$  line is flat from  $x = -1/2$  to  $x = 1/2$ .

### A. Mesonic hybrid potential

For the  $q\bar{q}$  system the arguments of the previous section lead to a non-relativistic potential that is a standard linearly-rising potential for each of the areas 1G and G2 in Fig. 1:

$$V(\vec{R}, \vec{r}) = K_F \{ [(z - R/2)^2 + \rho^2]^{1/2} + [(z + R/2)^2 + \rho^2]^{1/2} - R \} \quad (19)$$

[This potential has been proposed before [16] on purely phenomenological grounds.] This potential has all the properties expected from our discussion in the previous section. In the minimal surface of the  $q\bar{q}$  system, which is  $\rho = 0$ ,  $|z| < R/2$ , the potential is identically zero. For  $|z| > R/2$  there are linearly-rising terms beginning at  $|z| = R/2$  and extending outside the minimal surface, as explained in the Appendix. We identify these terms with the breakable adjoint string, although the potential of Eq. (19) does not explicitly show breaking. For small excursions in  $\rho$  (transverse to the minimal surface) we expand the potential in  $\rho$ . The linear terms vanish, as they must (except at the endpoints  $|z| = R/2$ ), and near  $z \simeq 0$  the quadratic terms are of the form  $(2K_F\rho^2/R)$ , with an oscillator frequency  $\Omega(R) \simeq 2[K_F/(MR)]^{1/2}$ , where  $R$  is the  $q\bar{q}$  separation. As  $z$  approaches the endpoints  $\pm R/2$ , the energy needed for a given transverse displacement grows, so there is a force keeping the gluon “bead” away from the endpoints unless the string on which the bead rides is not stretched at all. In Fig. 5 we plot the potential for the  $d = 3$  case, with  $q$  and  $\bar{q}$  separated by unit distance along the  $x$  axis and  $y$  as the transverse coordinate.

For further discussion of this potential, see the Appendix.

### B. Baryonic hybrid potential

In general, the potential is the string tension  $K_F$  times the length of some minimal string network joining the original three quarks to the  $q\bar{q}$  pair of the gluon. We will see that in principle this potential requires consideration of minimal string lengths with different functional forms in different regimes. Once the work of expanding the potential in various regimes is all done, we get a result which differs only slightly from the result gotten from the intuitively-appealing potential

$$V(\vec{R}_1, \vec{R}_2, \vec{R}_3; \vec{r}) = K_F \{ |\vec{R}_1 - \vec{r}| + |\vec{R}_2 - \vec{r}| + |\vec{R}_3 - \vec{r}| \}. \quad (20)$$

The reader who is willing to believe this can go straight to the end of this section, where the final results for the harmonic terms are given. We will only work out the case when the three quarks (at fixed time) are at the corners of an equilateral triangle. This makes little difference in comparing theory with data, since [5] the corresponding lattice data are not much different for different quark triangles, provided that they have the same length  $L_{min}$  of the Y-shaped Steiner string network joining them.

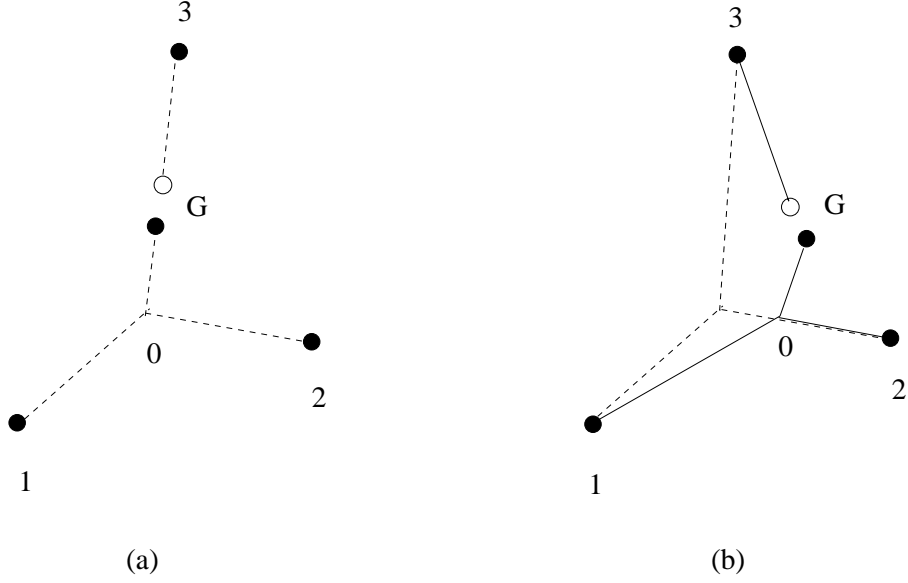


FIG. 6: A view of the  $qq\bar{q}G$  system looking along the quark world lines. The original Steiner network for no gluon is shown as dotted lines meeting at 0. (a) The gluon lies along the line 03; there is no resulting potential (except at the end points). (b) The gluon lies in regime 1 (defined in the text); the original Y strings are deformed into a longer Steiner network, raising the potential from zero.

Fig. 6 shows a view of a  $qq\bar{q}G$  system looking down along the  $t$  axis on the transverse plane, taken to be the  $xy$  plane. The original quark system 123 has no angle as great as  $2\pi/3$  and so has an internal Steiner point; this is the only type of configuration we will consider here. Full circles represent the quarks 1, 2, 3 of Fig. 2 and the quark in the split gluon line, and the unfilled circle is the antiquark of that line. Dotted lines represent the original Y-shaped strings joining the quarks in the absence of the gluon; generically they meet at the point 0 in Fig. 6(a) at an angle of  $2\pi/3$ . Splitting the gluon leads to a new baryon 12 $\bar{G}$ , which is joined by its own string network.

We have chosen to single out quark 3 for special treatment in forming the baryonic hybrid. This could be appropriate if the quarks had different flavors or masses. Later, we will average the potential over interchanges of the three quarks to yield an energy symmetric in the quark labels. This is equivalent to dropping certain  $l = 2$  terms in the potential.

The essential point to keep in mind for the following discussion is that if a triangle has no angle as great as  $2\pi/3$ , it has a Steiner point inside the triangle where the three strings meet at an angle of  $2\pi/3$ . But if an angle of the triangle is  $2\pi/3$  or greater, the minimal-length string lies along the two sides meeting at that angle. Because of this, the algebraic form of the potential changes, depending on which of several regimes the gluon occupies. Fluctuations of the gluon in the out-of-plane ( $z$ ) direction decouple from fluctuations in the  $XY$  plane, and are easily treated separately. If the gluon is fluctuating in-plane, we need to consider two regimes. Regime 1 is the sum of triangles 103 and 203 in the figure; regime 2 is the triangle 012.

Consider first regime 1. For the special position shown in Fig. 6(a), with the gluon sitting on the original Steiner line 03, the strings for the new baryon coincide with those of the original. This is because the angle 102 is precisely  $2\pi/3$ . This, plus our remarks already made for a gluon lying in a mesonic surface, show that there is no potential energy associated with moving the gluon along the line 03, except within a QCD length of the ends 0 and 3. Let us take the line 03 to lie in the  $y$  direction; then we expect to (and do) find only oscillations in the  $x$  direction.

The potential to be expanded for all of regime 1 (again referenced to the original  $qq\bar{q}$  potential) is

$$V(\vec{R}_1, \vec{R}_2, \vec{R}_3; \vec{r}) = K_F \{L_{\min}(R_{1G}, R_{2G}, R_{12}) + |\vec{R}_3 - \vec{r}| - L_{\min}(R_{12}, R_{13}, R_{23})\} \quad (21)$$

where  $R_i$  is the vector position of the  $i^{th}$  quark,  $\vec{r}$  is the in-plane position of the gluon,  $R_{ij}$  is the distance between quarks  $i$  and  $j$ , and  $R_{iG}$  the distance between quark  $i$  and the gluon. Also,  $L_{\min}(a, b, c)$  is the length of the Steiner network joining three points that form a triangle of sides  $a, b, c$ . The expression for this minimal length is:

$$L_{\min}(a, b, c) = \frac{1}{\sqrt{2}} \{(a^2 + b^2 + c^2) + \sqrt{3}[4a^2b^2 - (a^2 + b^2 - c^2)^2]^{1/2}\}^{1/2}. \quad (22)$$

We take  $\vec{\rho} = (x, 0)$  and find the quadratic term in the potential

$$\frac{9}{4} \frac{K_F x^2}{L_{min}} \quad (23)$$

where  $L_{min}$  is the minimal length of the Steiner network joining the original three quarks. [If we had expanded the intuitive potential of Eq. (20) we would have found an equal contribution for  $y$  oscillations.] We will replace this by an average over choosing any of the quarks as having the gluon antiquark joined to it, instead of quark 3 as in the figure. This just amounts to isotropizing the result of Eq. (23) in the  $x - y$  plane, with a resulting contribution from regime 1 to the potential of

$$V_1 = \frac{9}{8} \frac{K_F}{L_{min}} (x^2 + y^2). \quad (24)$$

This isotropization drops some  $l = 2$ ,  $m = 2$  terms in the potential.

Regime 2 is the triangle 012 of Fig. 6(a). If the gluon fluctuates (in-plane) into this triangle, the baryon triangle  $1G2$  has an angle greater than  $2\pi/3$ . As a result, the minimal network is the sum of strings running from the original quarks to the gluon, and the potential is indeed that of the simple intuitive form in Eq. (20). This potential has the property that it is symmetric with respect to interchange of quark labels. It has an equilibrium point at the origin, reflecting the nature of the origin as a Steiner point.

We give the expansion of the regime-2 potential for generic interquark distances, for future studies. Choose the (two-dimensional) quark vectors as

$$\vec{R}_1 = R_1 \left( \frac{-\sqrt{3}}{2}, \frac{-1}{2} \right); \quad \vec{R}_2 = R_2 \left( \frac{\sqrt{3}}{2}, \frac{-1}{2} \right); \quad \vec{R}_3 = R_3 (0, 1). \quad (25)$$

With this coordinatization the Steiner point 0 is at the origin. The expansion of the potential of Eq. (21) in transverse coordinates is

$$V_2(\vec{R}_1, \vec{R}_2, \vec{R}_3; \vec{\rho}) = \frac{K_F}{2} \left\{ \rho^2 \sum \frac{1}{R_i} - \sum \frac{[\hat{R}_i \cdot \vec{\rho}]^2}{R_i} \right\} + \dots \quad (26)$$

where  $\vec{\rho} = (x, y)$ . Working out the algebra, using Eq. (25), yields

$$V_2(\vec{R}_1, \vec{R}_2, \vec{R}_3; \vec{r}) = \frac{K_F}{2} \rho_i V_{ij} \rho_j \quad (27)$$

where the matrix  $V_{ij}$  has eigenvalues  $\lambda_\alpha$ :

$$\lambda_1, \lambda_2 = \frac{1}{2} \left\{ \sum \frac{1}{R_i} \pm \left[ \sum \frac{1}{R_i^2} - \sum_{i \neq j} \frac{1}{R_i R_j} \right]^{1/2} \right\}. \quad (28)$$

This energy  $\varepsilon(R_1, R_2, R_3)$  is clearly not a function of the minimal distance  $L_{min} = R_1 + R_2 + R_3$ . The deviations from dependence only on  $L_{min}$  are acceptably small as long as none of the  $R_i$  are too small, which in any case is necessary for all of our developments. For the equilateral triangle, the regime-2 potential is

$$V_2 = \frac{9}{4} \frac{K_F}{L_{min}} (x^2 + y^2). \quad (29)$$

Finally, the out-of-plane oscillations come from expanding Eq. (20) for a gluonic excursion in  $z$ . The result is

$$V_z = \frac{9}{2} \frac{K_F}{L_{min}} z^2. \quad (30)$$

To construct the final potential, we note that regime 1 is twice the size of regime 2. For a simple harmonic-oscillator wave function as a variational trial wave function this suggests that we take the in-plane potential as

$$V_{in} = \frac{2}{3} V_1 + \frac{1}{3} V_2 = \frac{3}{2} \frac{K_F}{L_{min}} (x^2 + y^2). \quad (31)$$

The total potential is

$$V_{in} + V_z \equiv V(\vec{x}) = \frac{\beta_z K_F}{L_{min}} z^2 + \frac{\beta_\rho K_F}{L_{min}} (x^2 + y^2) \quad (32)$$

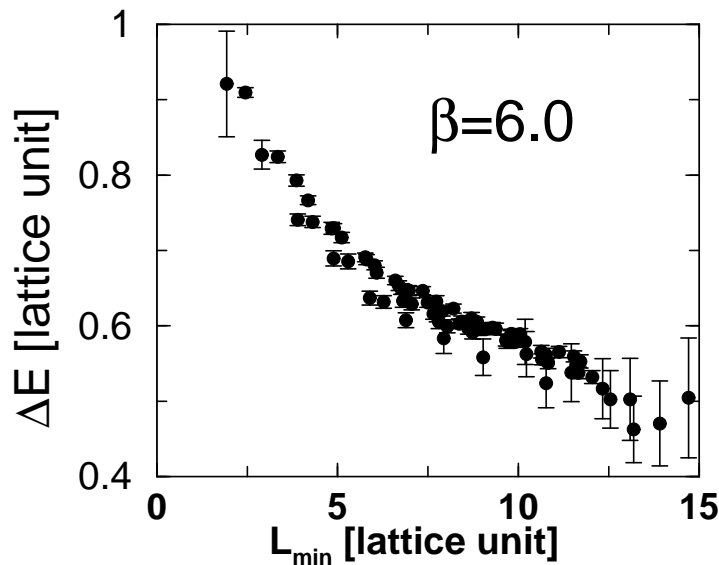


FIG. 7: Lattice determination of the single-gluon excitation energy  $\Delta E$  of a heavy  $qq\bar{q}G$  system, as a function of  $L_{min}$ , the length of the Steiner path joining the three quarks. From [5].

where  $\beta_z = 9/2$ ,  $\beta_\rho = 3/2$ . This potential can be isotropized in  $z$  but it makes little numerical difference.

Strictly speaking, the harmonic potentials in  $V$  terminate at the positions of the quarks in the original baryon, and are replaced by something more complicated (essentially the breakable adjoint string). However, in the regime of large interquark separations this is not important. The reason is that if the unadorned harmonic potentials are used, the (non-relativistic) range of the gluon wave function is of order  $[L_{min}/(MK_F)]^{1/4}$ . This is small compared to  $L_{min}$  when  $L_{min} \gg a$ , with  $a = (2K_F M)^{-1/3} \simeq 0.33$  f. Even if this termination of harmonic potentials is taken into effect, it is a small effect numerically.

The non-relativistic ground-state energy of the total potential  $V$ , plus the gluon mass  $M$ , yields a string energy for the baryon hybrid of

$$\varepsilon_{string}(L_{min}) = M + \left(\frac{3}{2} + \sqrt{3}\right) \left[\frac{K_F}{ML_{min}}\right]^{1/2} \equiv \xi \left[\frac{K_F}{ML_{min}}\right]^{1/2}; \quad \xi \simeq 3.23. \quad (33)$$

To this we add the color-Coulomb energy difference, estimated in the Appendix as

$$V_c = \alpha_c \left\{ \frac{\sqrt{3}}{L_{min}} - \frac{6}{\sqrt{L_{min}^2 + 9a^2}} \right\} \quad (34)$$

with  $\alpha_c \simeq 0.15$  [5] and  $a$ , defined above, is  $\simeq 0.33$  f. (The length  $a$  accounts for spreading of the gluon wave function, as described in the Appendix.)

#### IV. COMPARISON OF THEORY WITH LATTICE DATA

There are extensive heavy-quark lattice data available on the heavy-quark  $qq\bar{q}G$  system [5]. Fig. 7 shows the measured gluonic excitation energy  $\Delta E$  of the gluon in the  $qq\bar{q}G$  system, measured relative to the  $qq\bar{q}$  system, and displayed as a function of the length of strings joining the original three quarks [5]. Although in principle the energy depends on three separate interquark distances, [5] finds that their lattice results fall pretty much on a single curve, parametrized by the minimum length  $L_{min}$  of the Y-string joining the original  $qq\bar{q}$  system. Both the energy and distance axes are scaled in lattice units, which for [5] is about 0.1 fm. The energies in GeV are approximately twice the numerical values shown on Fig. 7 and on the next figure.

We can match the data of Fig. 7 reasonably well, using the variational treatment of the anisotropic harmonic-oscillator Schrödinger equation with relativistic kinetic energy, given in the Appendix, and a gluon mass  $M$  of around 500 MeV. Coulomb effects are added perturbatively, as described in the Appendix. Since the data themselves fall roughly on a single curve for various  $qq\bar{q}$  geometries with the same  $L_{min}$ , we have done numerical calculations only

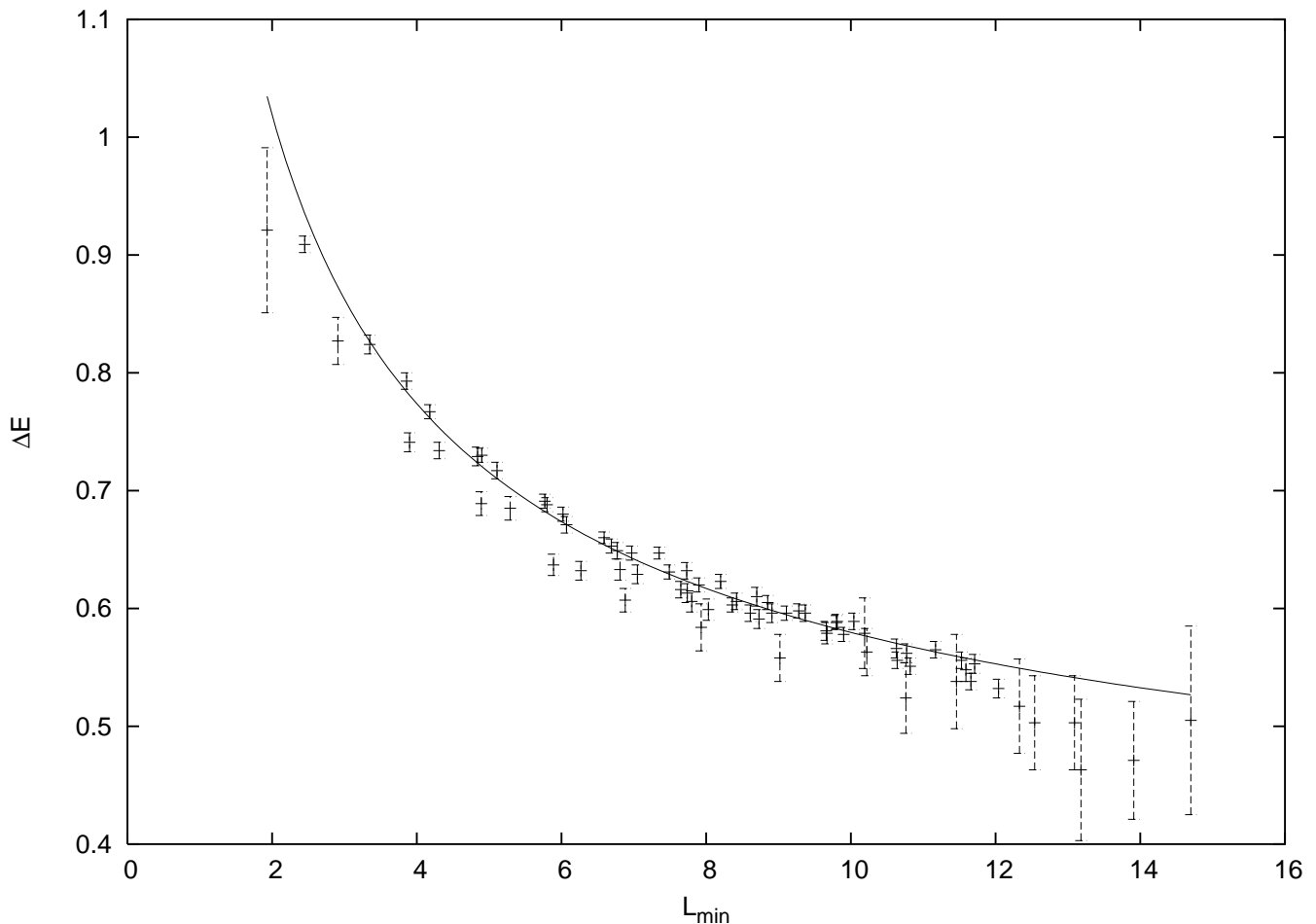


FIG. 8: Data (again in lattice units) of Fig. 7 shown with the theoretical calculation discussed in the text.

for the special case of an equilateral triangle. These calculations and the lattice data [5] are shown in Fig. 8 for the value  $M = 500$  MeV.

The fit is surprisingly good at  $L_{min} \geq 4$ , which is a distance short enough for relativistic effects to be important. In fact, the total energy at this distance is about  $3M$ , roughly apportioned equally among mass energy, potential energy, and kinetic energy. For smaller  $L_{min}$  there really should be a fair amount of mixing with states containing more than one gluon, such as a baryon plus a glueball. We do not know to what extent such mixing is or is not accounted for in the lattice computations. The mass value of 500 MeV is a little small compared to a typical lattice value [8] of 600 MeV, but certainly falls within an acceptable range of determinations of  $M$ . A proper handling of mixing of higher angular momentum states might raise the fitted gluon mass somewhat.

## V. CONCLUSIONS

We showed that there is a straightforward path from the fundamental elements of center-vortex QCD to a description of the gluonic ground state of heavy-quark  $qqqG$  hybrids (and, for that matter, of  $q\bar{q}$  hybrids, although we have not attempted any comparison with lattice data for these). In graphic terms, the result is that the valence gluon can be thought of as a massive bead more or less freely running along a string joining the quarks, and subject to transverse restoring forces as the string is stretched by gluonic fluctuations.

The hybrid problem simplifies at large interquark separation  $R$ , which is the regime we treated explicitly. Although our treatment, based on approximations to a Schrödinger equation with fully-relativistic kinetic energy is not exact, it is not subject to the usual burden of purely phenomenological assumptions made about the nature of confining potentials and so forth. The fundamental elements used were: 1) Splitting of a gluon proper-time adjoint Wilson line into a pair of co-located and oppositely-directed fundamental Wilson lines, which join to the three other Wilson

lines in a specific way. 2) Recognition that for a massive gluon the equal-time Schrödinger equation applies for sufficiently large  $R$ , but that relativistic corrections to the kinetic energy are important for physically-interesting quark separations.

We used the gluon mass  $M$  as a single parameter for fitting the lattice data of Ref. [5], and found a pretty good fit for a mass of about 500 MeV. This mass is a little less than the usual value of about 600 MeV found on the lattice, but at this stage of the investigation seems quite satisfactory.

One could, in principle, think of extending this work to excited gluonic states in hybrids. However, at any reasonable interquark separation, the excited states are relativistic, and there will be some mixing with states with more than one gluon. As discussed in the Appendix for  $q\bar{q}G$  hybrids, if this mixing is ignored there should be several contributions to the gluonic energy, including a stringlike energy of the form  $N\pi/R$ , as invoked by [6, 12] for such hybrids, plus terms of order  $(K_F/R)^{1/3}$  and Coulomb terms.

We noted that the treatment of the valence gluon by adjoint-line splitting suggested a possible path toward describing the fluctuations of surfaces spanning fundamental Wilson loops. Originally, these surfaces were envisaged [13] as having an essentially tangible existence as chromoelectric flux tubes. But fluctuations of spanning surfaces in the area law is more difficult to think about in the center vortex picture, where the description of confinement begins with the topological statement that the area law arises from the fluctuations in phase factors (elements of the center group) associated with topological linking of center vortices with the Wilson loop. At this level of description of confinement, the area in an area law is a minimal surface spanning the Wilson loop and is not subject to fluctuations [9]. However, coupling of gluons that are not part of the condensate of confining vortices can give rise to surface fluctuations, as we have seen in the present paper for valence gluons in hybrids. Work is underway to understand the description of such fluctuations when a large number of such gluons are coupled to a Wilson loop.

Finally, we have not discussed here the mixing of the valence gluon in a hybrid with genuine  $q\bar{q}$  pairs (as opposed to the fictitious pairs produced by splitting an adjoint Wilson line). If the valence gluon in  $qqqG$  is replaced by a valence  $q\bar{q}$  pair, the result is a special kind of pentaquark, or  $qqqq\bar{q}$  system. This pentaquark is definitely not the same as the famous pentaquark [17], which is of the form  $uudd\bar{s}$ , where the antiquark cannot pair with any quark to form a flavor-neutral state such as a gluon. Nonetheless, the problem of baryonic hybrid mixing with pentaquarks is of some interest, and worthy of future study.

### Acknowledgments

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## APPENDIX A: TREATMENT OF THE SCHRÖDINGER EQUATION

Although the gluon has a mass  $M$ , it is not heavy enough to insure that the non-relativistic Schrödinger equation applies throughout the range of quark separations for which lattice data are available. We give first a simple variational treatment of the ground-state excitation energy associated with the harmonic potentials encountered for hybrids, including relativistic effects for the kinetic energy. Then we go on to sketch a treatment of the gluon as a bead sliding freely along a string, and confined harmonically in the transverse direction. This is the problem encountered for the  $q\bar{q}G$  hybrid and for one of the strings in the  $qqqG$  hybrid. Finally, we briefly sketch a treatment for color-Coulomb corrections.

### 1. Relativistic harmonic oscillator for the baryonic hybrid

The string potential for the  $qqqG$  state is the sum of Eqs. (30, 31), and has the form

$$V(\vec{x}) = \frac{\beta_z K_F}{L_{min}} z^2 + \frac{\beta_\rho K_F}{L_{min}} (x^2 + y^2) \quad (A1)$$

where  $\beta_z = 9/2$ ,  $\beta_\rho = 3/2$  and  $L_{min}$  is the minimum length of the Steiner network. The full Hamiltonian, including relativistic kinetic energy, is

$$H = \sqrt{\vec{p}^2 + M^2} + V(\vec{x}). \quad (A2)$$

We take a simple normalized variational wavefunction  $\psi$ :

$$\psi(\vec{x}) = \left(\frac{\alpha_z}{\pi}\right)^{1/4} \left(\frac{\alpha_\rho}{\pi}\right)^{1/2} \exp\left\{-\frac{1}{2}[\alpha_\rho(x^2 + y^2) + \alpha_z z^2]\right\} \quad (A3)$$

and calculate the expectation value  $\langle H \rangle \equiv \varepsilon(\alpha_z, \alpha_\rho)$ :

$$\langle H \rangle = \frac{2}{\alpha_\rho \sqrt{\alpha_z \pi}} \int_0^\infty dp_\perp p_\perp \int_0^\infty dp_\parallel (p_\perp^2 + p_\parallel^2 + M^2)^{1/2} \exp\left\{-\frac{p_\parallel^2}{\alpha_z} - \frac{p_\perp^2}{\alpha_\rho}\right\} + \frac{K_F}{2L_{min}} \left(\frac{\beta_z}{\alpha_z} + \frac{2\beta_\rho}{\alpha_\rho}\right). \quad (\text{A4})$$

The variational parameters  $\alpha_{\rho,z}$  are to be determined by minimizing  $\varepsilon$ , which can only be done numerically. Of course, the non-relativistic limit is easy to do, yielding

$$\langle H \rangle = M + \left(\frac{K_F}{2ML_{min}}\right)^{1/2} [\sqrt{\beta_z} + 2\sqrt{\beta_\rho}]. \quad (\text{A5})$$

In the relativistic regime, one finds

$$\varepsilon = \frac{3}{2} \left(\frac{3\beta K_F}{2\pi L_{min}}\right)^{1/3}. \quad (\text{A6})$$

where  $\beta_{rel}$  is a weighted average of  $\beta_z$  and  $\beta_\rho$ , approximately given by  $\beta_{rel} = (1/3)(\beta_z + 2\beta_\rho)$ . The separation between the relativistic and non-relativistic regimes is at a critical interquark distance  $R_c$ , where

$$R_c \sim \frac{\beta K_F}{M^3}. \quad (\text{A7})$$

When  $R \gg R_c$  the problem is non-relativistic. Typically  $R_c$  is perhaps 0.8 f.

The fit to data of Fig. 8 comes from a combination of numerical and analytical work on minimizing expressions such as in Eq. (A4). It turns out to be accurate enough to isotropize the potential of Eq. (A1, setting  $x^2$ ,  $y^2$  and  $z^2$  all to  $r^2/3$ , and also to take  $\alpha_z = \alpha_\rho \equiv \alpha$ . Then a good analytic fit can be made to the kinetic energy term, and the result is

$$\varepsilon(\alpha) \simeq [\gamma\alpha + M^2]^{1/2} + \frac{3\beta K_F}{2\alpha R} \quad (\text{A8})$$

with

$$\gamma \simeq 1.38, \quad \beta \simeq 2.52. \quad (\text{A9})$$

The values of  $\gamma$  and  $\beta$  are correlated so as to give the exact non-relativistic energy at large  $R$ . It is now straightforward if lengthy to find an analytic expression for the minimum over  $\alpha$  of  $\varepsilon(\alpha)$ , involving the root of a quartic equation.

## 2. Line of equilibrium

We continue with a treatment of the Schrödinger equation that applies when there is a line of equilibrium, not just a point.

To be specific, consider a  $q\bar{q}$  state, where the potential has the form in Eq. (19) (repeated for convenience)

$$V(\vec{R}, \vec{r}) = K_F \{[(z - R/2)^2 + \rho^2]^{1/2} + [(z + R/2)^2 + \rho^2]^{1/2} - R\} \quad (\text{A10})$$

. Here (see Fig. 1)  $\vec{R} = (0, 0, R)$  is the separation vector of the heavy quark and antiquark,  $\vec{r} = (x, y, z)$  is the gluon position vector, and  $\rho^2 = x^2 + y^2$  is the (square of the) gluon excursion out of the minimal surface. As discussed earlier, this is not literally the true potential, but it is accurate enough for our purposes.

There is a second scale length  $a$  in the problem, given by

$$a = (2K_F M)^{-1/3} \simeq 0.33 fm, \quad (\text{A11})$$

and we will require  $R \gg a$ . Since  $R_c/a \sim K_F/M^2 < 1$ , this requirement puts us in the non-relativistic regime, and means that our considerations may hold for the ground state, but not necessarily for excited states.

We will see that the scale length for gluon excursions away from the minimal  $q\bar{q}$  surface is at most of order  $(Ra^3)^{1/4} \ll R$ . Therefore we provisionally take  $\rho^2/R^2$  as an expansion parameter for the potential. Such an expansion requires  $|z| \ll R/2$ , which we assume for now. The first two terms of the expansion are

$$V = \frac{2K_F \rho^2}{R} + \frac{8K_F z^2 \rho^2}{R^3} + \dots \quad (\text{A12})$$

There are no terms in the expansion of the form  $z^N$ ; all powers of  $z$  are accompanied by powers of  $\rho^2$ . Note that if  $\rho \neq 0$ , the point  $z = 0$  is the equilibrium point, midway between the quark and antiquark.

The expansion above fails for  $|z| > R/2$ . To characterize this regime we set  $\rho = 0$  in the potential, and find that the potential identically vanishes for  $|z| < R/2$ , while for  $|z| > R/2$  there are linearly-rising potentials which confine the  $z$  motion. We will adopt the following essentially variational strategy, accurate enough for present purposes: Assume the ground-state wave function  $\psi(\vec{r})$  for the Schrödinger equation is separable, of the form

$$\psi = \phi(z) \left(\frac{\alpha}{\pi}\right)^{1/2} \exp\left[-\frac{\alpha \rho^2}{2}\right] \quad (\text{A13})$$

where  $\phi(z)$  is a normalized solution to the Schrödinger equation at  $\rho = 0$ . The other factor is a harmonic-oscillator ground state, and we will determine the parameter  $\alpha$  from the potential of Eq. (A10), averaged over the  $z$  motion. That is, the effective potential for  $\rho$  is given by

$$V_{eff}(\rho) = \int_{-R/2}^{R/2} dz |\phi(z)|^2 V(\rho, z). \quad (\text{A14})$$

We now construct a good approximation to  $\phi(z)$ . For  $z > 0$  the potential is accurately given by

$$V(z) \simeq 2K_F |z - (R/2)| \theta(z - (R/2)), \quad (\text{A15})$$

and the region  $z < 0$  contributes symmetrically. The coefficient  $2K_F$  is a decent approximation to the true adjoint string tension. Of course, the adjoint string is breakable, and the potential of Eq. (A15) is applicable only for sufficiently small  $|z - (R/2)|$ . The ground-state solution to the Schrödinger equation which is symmetric under  $z \rightarrow -z$  is

$$\phi(z) = N_1 \cos(qz) \theta[(R/2) - |z|] + \{N_2 Ai(\xi) \theta[z - (R/2)] + z \leftrightarrow -z\}. \quad (\text{A16})$$

Here  $N_{1,2}$  are normalization constants,  $q$  is the momentum, and  $Ai(\xi)$  is an Airy function of argument

$$\xi = \frac{1}{a} \left(z - \frac{R}{2}\right) - q^2 a^2. \quad (\text{A17})$$

The  $z$ -motion energy eigenvalue is  $\varepsilon_z = q^2/(2M)$ , with the momentum  $q$  determined by the matching condition

$$-qa \tan\left(\frac{qR}{2}\right) = \frac{Ai'(\xi)}{Ai(\xi)} \Big|_{\xi=-(qa)^2}. \quad (\text{A18})$$

At  $R = 0$  this yields the well-known result  $(qa)^2 \simeq \xi_0$  where  $-\xi_0 = -1.02+$  is the first zero of  $Ai'(\xi)$ . For large  $R$  one easily sees that  $q$  must decrease toward  $\pi/R$ , so that  $\varepsilon_z \simeq \pi^2/(2MR^2)$ . Analysis which we do not give here suggests that the following form should be a good representation of the momentum  $q(R)$  and  $\varepsilon_z(R)$  as functions of  $R$ :

$$q(R) \simeq \frac{\xi_0^{1/2}}{a + \gamma R}; \quad \varepsilon_z(R) \simeq \frac{\xi_0}{2Ma^2(1 + \gamma R/a)^2} \quad (\text{A19})$$

where  $\gamma = \xi_0^{1/2}/\pi$ . This correctly yields the large- $R$  behavior and is correct to about 20% for small  $R$ .

After some uninteresting algebra, one finds that at large  $R$  the Airy functions' renormalization constant  $N_2$  is small of order  $a/R$ , and therefore in calculating the  $\rho$  potential from Eq. (A14) we need the integral

$$V_{eff}(\rho) = K_F \int_{-R/2}^{R/2} dz \frac{2}{R} \cos^2(qz) \{[(z - R/2)^2 + \rho^2]^{1/2} + [(z + R/2)^2 + \rho^2]^{1/2} - R\}. \quad (\text{A20})$$

We have calculated this integral numerically, and find that for small  $(\rho/R)^2$  it is reasonably well-approximated by the  $z = 0$  expansion already given in Eq. (A12):  $V_{eff} \simeq 2K_F \rho^2/R$ . The reason is that in the integral the weight function  $\cos^2(qz)$  is small near the end points. Now the parameter  $\alpha$  of the transverse wave function in Eq. (A13) is determined ( $\alpha = 2(K_F M/R)^{1/2}$ ), and the total non-relativistic energy at large  $R$  is

$$\varepsilon(R) \simeq M + 2\left(\frac{K_F}{MR}\right)^{1/2} + \frac{2\xi_0}{2Ma^2(1 + \gamma R/a)^2}. \quad (\text{A21})$$

For this oscillator one has  $\langle \rho^2 \rangle \simeq (Ra^3)^{1/2}$  which, as mentioned above, justifies expanding the full potential in powers of  $(\rho/R)^2$  at large  $R$ .

Excited-state solutions to the gluon motion in the  $z$  direction have momentum  $q$  given by  $\pi N/R$ . When  $N > MR/\pi$  the  $z$  motion is relativistic, with energy approximately equal to  $q$ . The transverse motion may be relativistic as well, requiring consideration of string breaking by gluon-pair formation. Without taking this phenomenon into account, one may expect corrections to the string energy  $N\pi/R$  to scale [as given in Eq. (A6)] with  $R^{-1/3}$ . However, many other effects come into play, and we have not attempted to analyze data, such as that of [12].



### 3. Color-Coulomb corrections

We will treat the Coulomb potential as a perturbation to the Schrödinger equation. If this potential were uniformly of order  $\alpha_c/R$  for typical interquark separation  $R$ , treating it as a perturbation would be valid (non-relativistically) when

$$R \gg \frac{\alpha_c^2 M}{K_F}. \quad (\text{A22})$$

which is easily satisfied for  $R$  values of concern to us.

#### a. Baryonic hybrid

In the baryonic hybrid  $qqqG$ , the minimum of the gluon potential is at the Steiner point, so it does not get close to the quarks, at least when these are well-separated ( $R \gg a$ ). When the quarks get close, so that  $R \leq a$ , the potential ceases to be  $1/R$ , due to smearing by the gluon wave function. The net color-Coulomb potential for the quarks, which can be treated classically, in the original baryon is to be subtracted from the Coulomb potential for the hybrid. To a good enough approximation, the net Coulomb potential for our standard equilateral triangle is

$$V_c = \alpha_c \left\{ \frac{\sqrt{3}}{L_{min}} - \frac{6}{\sqrt{L_{min}^2 + 9a^2}} \right\}. \quad (\text{A23})$$

We will use  $\alpha_c \simeq 0.15$ , appropriate for the lattice with  $\beta = 6.0$  and quenched quarks.

#### b. Mesonic hybrid

In the mesonic hybrid the gluon can get close to a quark even when these are well-separated, and the above argument does not straightforwardly apply when the gluon position  $z$  along the string is near  $\pm R/2$ , where the distance scale in the gluon-quark Coulomb potential is not generically of order  $R$ . Nonetheless, we will find that the probability that the gluon approaches the ends of its string, where the Coulomb potential is not of order  $\alpha_c/R$ , scales with  $1/R$  so that in the end the Coulomb contribution to the energy does scale with  $1/R$ .

The Coulomb potential  $V_C$  (relative to the  $q\bar{q}$  potential with no gluon) is

$$V_C = \alpha_c \left\{ \frac{1}{R} - [(z - R/2)^2 + \rho^2]^{-1/2} - [(z + R/2)^2 + \rho^2]^{-1/2} \right\}. \quad (\text{A24})$$

A reasonable approximation is to replace the terms  $z \pm (R/2)^2$  in  $V_C$  by an expectation value of order  $a^2$ , to replace  $\rho^2$  by an expectation value of order  $(R/(MK_F))^{1/2} \sim a^2(R/a)^{1/2}$ , and then to multiply the result by the probability that the gluon is within a distance  $a$  of  $z = \pm R/2$ . This probability is of order  $a/R$  at large  $R$ , as one sees from the matching of the wave functions given in Eq. (A18). Of course, when  $z \ll R$  the Coulomb potential is of order  $\alpha_c/R$ . The upshot is that all terms in the Coulomb potential are of this order:

$$V_C \simeq \frac{-A_1 \alpha_c}{R[1 + (R/a)^{1/2}]^{1/2}} - \frac{A_2 \alpha_c}{[R^2 + a^2(R/a)^{1/2}]^{1/2}} + \frac{\alpha_c}{R}. \quad (\text{A25})$$

where  $A_{1,2}$  are positive coefficients of order unity. Since we are not treating the mesonic hybrid in detail, we have neglected to write some other factors of order one at various other places in this equation.

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- [1] Sometimes the term “exotic” is restricted to states with extra valence quarks beyond  $q\bar{q}$  or  $qqq$ ; see R. L. Jaffe, arXiv:hep-ph/0409065 (October 2004).
  - [2] J. M. Cornwall and S. F. Tuan, Phys. Lett. B **136**, 110 (1984).
  - [3] N. Isgur and J. Paton, Phys. Rev. D **31**, 2910 (1985).
  - [4] C. A. Meyer, AIP Conf. Proc. **698**, 554 (2004) [arXiv:hep-ex/0308010].
  - [5] T. T. Takahashi and H. Suganuma, arXiv:hep-lat/0409105 (September 2004).
  - [6] K. J. Juge, J. Kuti and C. Morningstar, AIP Conf. Proc. **688**, 193 (2004) [arXiv:nucl-th/0307116].
  - [7] J. M. Cornwall, Phys. Rev. D **26**, 1453 (1982).
  - [8] C. Alexandrou, P. de Forcrand and E. Follana, Phys. Rev. D **65**, 114508 (2002) [arXiv:hep-lat/0112043].
  - [9] J. M. Cornwall, Phys. Rev. D **69**, 065019 (2004) [arXiv:hep-th/0304182].
  - [10] J. M. Cornwall, Phys. Rev. D **69**, 065013 (2004) [arXiv:hep-th/0305101].
  - [11] Y. A. Simonov, arXiv:hep-ph/0406290, and earlier works cited therein.
  - [12] K. J. Juge, J. Kuti and C. Morningstar, Phys. Rev. Lett. **90**, 161601 (2003) [arXiv:hep-lat/0207004].
  - [13] M. Lüscher, Nucl. Phys. B **180**, 317 (1981); M. Lüscher, K. Symanzik and P. Weisz, Nucl. Phys. B **173**, 365 (1980).
  - [14] M. Faber, J. Greensite and Š. Olejník, Phys. Rev. D **57**, 2603 (1998) [arXiv:hep-lat/9710039].
  - [15] J. M. Cornwall, Phys. Rev. D **58**, 105028 (1998) [arXiv:hep-th/9806007].
  - [16] A. Le Yaouanc, L. Oliver, O. Pène, J. C. Raynal and S. Ono, Z. Phys. C **28**, 309 (1985).
  - [17] For a recent review of both theory and experiment on pentquarks, see Jaffe, ref. 1, and S. L. Zhu, arXiv:hep-ph/0410002.